

# Complex hyperbolic triangle groups with 2-fold symmetry

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**Abstract.** In this paper we will consider the 2-fold symmetric complex hyperbolic triangle groups generated by three complex reflections through angle  $2\pi/p$  with  $p \geq 2$ . We will mainly concentrate on the groups where some elements are elliptic of finite order. Then we will classify all such groups which are candidates for being discrete. There are only 4 types.

## 1. INTRODUCTION

A complex hyperbolic triangle is a triple  $(C_1, C_2, C_3)$  of complex geodesics in  $\mathbf{H}_{\mathbb{C}}^2$ . If each pair of complex geodesics intersects in  $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$  and the angles between  $C_{k-1}$  and  $C_k$  for  $k = 1, 2, 3$  (the indices are taken mod 3) are  $\pi/p_1, \pi/p_2, \pi/p_3$ , where  $p_1, p_2, p_3 \in \mathbb{N} \cup \{\infty\}$ , we call the triangle  $(C_1, C_2, C_3)$  a  $(p_1, p_2, p_3)$ -triangle. The intersection points of pairs of complex geodesics are called the *vertices* of the complex hyperbolic triangle. A group  $\Gamma$  is called a  $(p_1, p_2, p_3)$ -triangle group, if  $\Gamma$  is generated by three complex reflections  $R_1, R_2, R_3$  fixing sides  $C_1, C_2, C_3$  of  $(p_1, p_2, p_3)$ -triangle. Note that a complex reflection may have order greater than 2. In what follows we suppose that  $R_1, R_2$  and  $R_3$  all have order  $p \in \mathbb{Z}$  with  $p \geq 2$ .

Any two real hyperbolic triangle groups with the same intersection angles are conjugate in  $\text{Isom}^+(\mathbf{H}^2)$ , which is the orientation preserving isometry group of real hyperbolic plane, see section 10.6 in [1]. If we consider the groups in  $\text{PU}(2, 1) = \text{Aut}(\mathbf{H}_{\mathbb{C}}^2)$ , we will get the nontrivial deformations.

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The deformation theory of complex hyperbolic triangle groups was begun in [3] in which they investigated  $\Gamma$  of type  $(\infty, \infty, \infty)$  with  $p = 2$  (complex hyperbolic ideal triangle groups). Since then, there have been many developments referring to other types, such as [15, 6, 13] among which they mainly gave the necessary conditions of  $\Gamma$  to be discrete. Especially Parker and Paupert in [10] and [11] investigated the equilateral triangle group generated by three complex reflections with finite order. These include *Deraux's lattice*, *Livné's lattices*, *Mostow's lattices*. Our starting point is a result given by Thompson [14] where he investigated the non-equilateral triangle groups generated by three complex involutions (that is the order of the reflections is  $p = 2$ ). He obtained his result using a computer search. Using [11] we see that Thompson's results apply to groups with  $p > 2$  as well. In what follows we will give the specific case about the triangles group with 2-fold symmetry and we give a rigorous proof.

We will restrict to the complex hyperbolic triangle groups generated by three complex reflections with finite order  $p \geq 2$ . Suppose that the polar vector of a complex geodesic  $C_1$  is  $\mathbf{v}_1$  (see Section 2 for a more precise explanation). We consider the complex reflection  $R_1$  in the complex geodesic  $C_1$ . This map sends  $\mathbf{v}_1$  to  $e^{i\phi}\mathbf{v}_1$  and acts as the identity on the orthogonal complement of  $\mathbf{v}_1$ , that is on vectors that project to  $C_1$ . We will always restrict to the case where  $\phi = 2\pi/p$  and so  $R_1$  has order  $p \geq 2$ . Then  $R_1$  is given by the following formula:

$$R_1(\mathbf{z}) = \mathbf{z} + (e^{i\phi} - 1) \frac{\langle \mathbf{z}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1. \quad (1.1)$$

In order to convert  $R_1$  into a matrix with determinant 1, we need to multiply the expression in (1.1) by  $e^{-i\phi/3}$ . The ambiguity involved in this choice is precisely the ambiguity involved in lifting an isometry in  $\text{PU}(2, 1)$  to a matrix in  $\text{SU}(2, 1)$ .

Here we recall the terminology for *braid relations* between group elements (see Section 2.2 of Mostow [8]). Let  $G$  be a group and  $a, b \in G$ . Then  $a$  and  $b$  satisfy a braid relation of length  $l \in \mathbb{Z}_+$  if

$$(ab)^{l/2} = (ba)^{l/2},$$

where powers means that the corresponding alternating product of  $a$  and  $b$  should have  $l$  factors. For example,  $(ab)^{3/2} = aba$ ,  $(ba)^2 = baba$ . We denote the *braid length*  $l$  by  $br(a, b)$  to be the minimum length of a braid relation satisfied by  $a$  and  $b$ .

We define the  $(l_1, l_2, l_3; l_4)$ -triangle groups to be the triangle groups with the following braid relations:

$$\begin{aligned} br(R_2, R_3) &= l_1, & br(R_1, R_3) &= l_2, \\ br(R_1, R_2) &= l_3, & br(R_1, R_3^{-1}R_2R_3) &= l_4, \end{aligned}$$

where each  $R_j$  is of order  $p$ .

In this paper we aim to list the candidates of discrete triangle groups generated by  $R_1, R_2, R_3$  with  $l_1 = l_2$  and  $l_3 = l_4$  as stated in Theorem 2.4.

## 2. THE PARAMETER SPACE, TRACES AND MAIN RESULT

Firstly we recall some fundamentals about complex hyperbolic 2-space. Please refer to [4, 9] for more details about the complex hyperbolic space. Let  $\mathbb{C}^{2,1}$  denote the vector space  $\mathbb{C}^3$  equipped with the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3$$

of signature  $(2, 1)$ , where  $\mathbf{z} = [z_1, z_2, z_3]^t$  and  $\mathbf{w} = [w_1, w_2, w_3]^t$ . The Hermitian form divides  $\mathbb{C}^{2,1}$  into three parts  $V_-, V_0$  and  $V_+$ , which are

$$V_- = \{\mathbf{z} \in \mathbb{C}^{2,1} \mid \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$

$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} \mid \langle \mathbf{z}, \mathbf{z} \rangle = 0\},$$

$$V_+ = \{\mathbf{z} \in \mathbb{C}^{2,1} \mid \langle \mathbf{z}, \mathbf{z} \rangle > 0\}.$$

We denote by  $\mathbb{C}\mathbb{P}^2$  the complex projectivisation of  $\mathbb{C}^{2,1}$  and by

$$\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^2$$

the natural projectivisation map. The *complex hyperbolic 2-space*  $\mathbf{H}_{\mathbb{C}}^2$  is defined as  $\mathbb{P}(V_-)$ . It is called the *standard projective model* of the complex hyperbolic space. Correspondingly the boundary of  $\mathbf{H}_{\mathbb{C}}^2$  is  $\partial\mathbf{H}_{\mathbb{C}}^2 = \mathbb{P}(V_0 \setminus \{0\})$ . One can also consider the *unit ball model* whose boundary is the sphere  $\mathbb{S}^3$  by taking  $z_3 = 1$ , which can be simply written as

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}.$$

The complex hyperbolic plane  $\mathbf{H}_{\mathbb{C}}^2$  is a Kähler manifold of constant holomorphic sectional curvature  $-1$ . The holomorphic automorphism group of  $\mathbf{H}_{\mathbb{C}}^2$  is the projectivisation  $\mathrm{PU}(2, 1)$  of the group  $\mathrm{U}(2, 1)$  of complex linear transformations on  $\mathbb{C}^{2,1}$ , which preserve the Hermitian form. Especially  $\mathrm{SU}(2, 1)$  is the subgroup of  $\mathrm{U}(2, 1)$  with the determinant of each element being 1.

Let  $x, y \in \mathbf{H}_{\mathbb{C}}^2$  be points corresponding to vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{2,1} \setminus \{0\}$ . Then the *Bergman metric*  $\rho$  on  $\mathbf{H}_{\mathbb{C}}^2$  is given by

$$\cosh^2 \left( \frac{\rho(x, y)}{2} \right) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle},$$

where  $\mathbf{x}, \mathbf{y} \in V_-$  are the lifts of  $x, y$  respectively. It is easy to check that this definition is independent of the choice of lifts.

Given two points  $x$  and  $y$  in  $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ , with lifts  $\mathbf{x}$  and  $\mathbf{y}$  to  $\mathbb{C}^{2,1}$  respectively, the complex span of  $\mathbf{x}$  and  $\mathbf{y}$  projects to a *complex line* in  $\mathbb{C}\mathbb{P}^2$  passing through  $x$  and  $y$ . The intersection of a complex line with  $\mathbf{H}_{\mathbb{C}}^2$

will be called a *complex geodesic*  $C$  (which is homeomorphic to an open 2-dimensional disk), which can be uniquely determined by a positive vector  $\mathbf{v} \in V_+$ , i.e.  $C = \mathbb{P}(\{\mathbf{z} \in \mathbb{C}^{2,1} \setminus \{0\} \mid \langle \mathbf{z}, \mathbf{v} \rangle = 0\})$ . We call  $\mathbf{v}$  a *polar vector* to  $C$ . As stated in Section 1, we will consider  $(l_1, l_2, l_3; l_4)$ -triangle groups  $\Gamma$  generated by three complex reflections, see (1.1), through angle  $\phi$  in three complex geodesics.

Throughout this paper, we assume that  $R_1, R_2, R_3$  are three complex reflections in complex geodesics  $C_1, C_2, C_3$  respectively. We parameterize the triangle groups generated by  $R_1, R_2, R_3$  by three complex numbers  $\rho, \sigma$  and  $\tau$ . Up to the action of  $\text{PU}(2, 1)$ , we can parameterize the collection of three pairwise distinct complex lines in  $\mathbf{H}_{\mathbb{C}}^2$  by four real parameters, see Proposition 1 of [12]. The parameters we choose are  $|\rho|, |\sigma|, |\tau|$  and  $\arg(\rho\sigma\tau)$ . In particular, we can freely choose the argument of two out of the three parameters.

Write  $u = e^{i\phi/3} = e^{2\pi i/3p}$ . The group  $\Gamma$  has generators given by

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\bar{\tau} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sigma \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \tau & -u\bar{\sigma} & u^2 \end{pmatrix} \quad (2.1)$$

which preserve the Hermitian form

$$H = \begin{pmatrix} \alpha & \beta_1 & \bar{\beta}_3 \\ \bar{\beta}_1 & \alpha & \beta_2 \\ \beta_3 & \bar{\beta}_2 & \alpha \end{pmatrix}, \quad (2.2)$$

where  $\alpha = \sqrt{2 - u^3 - \bar{u}^3}$ ,  $\beta_1 = -i\bar{u}^{1/2}\rho$ ,  $\beta_2 = -i\bar{u}^{1/2}\sigma$ ,  $\beta_3 = -i\bar{u}^{1/2}\tau$  (note that here we take  $\bar{u}^{1/2} = e^{-\pi i/3p}$ ).

This Hermitian form has signature  $(2, 1)$  if and only if  $\det(H) < 0$ . That is,

$$\begin{aligned} 0 &< \alpha|\beta_1|^2 + \alpha|\beta_2|^2 + \alpha|\beta_3|^2 - \alpha^3 - \beta_1\beta_2\beta_3 - \bar{\beta}_1\bar{\beta}_2\bar{\beta}_3 \\ &= \alpha^2|\rho|^2 + \alpha^2|\sigma|^2 + \alpha^2|\tau|^2 - \alpha^3 - i\bar{u}^{3/2}\rho\sigma\tau + iu^{3/2}\bar{\rho}\bar{\sigma}\bar{\tau}. \end{aligned}$$

In terms of these parameters

$$\begin{aligned} \text{tr}(R_1R_2) &= u(2 - |\rho|^2) + \bar{u}^2, \\ \text{tr}(R_2R_3) &= u(2 - |\sigma|^2) + \bar{u}^2, \\ \text{tr}(R_1R_3) &= u(2 - |\tau|^2) + \bar{u}^2, \\ \text{tr}(R_1R_3^{-1}R_2R_3) &= u(2 - |\sigma\tau - \bar{\rho}|^2) + \bar{u}^2. \end{aligned} \quad (2.3)$$

**Lemma 2.1.** [11, Corollary 2.5] *If  $|\rho| = 2 \cos \zeta$ , then the three eigenvalues of  $R_1R_2$  will be  $\bar{u}^2, -ue^{2i\zeta}, -ue^{-2i\zeta}$ .*

**Proof.** Each point on  $C_1$  is a  $\bar{u} = e^{-i\phi/3}$  eigenvector of  $R_1$  and each point on  $C_2$  is a  $\bar{u} = e^{-i\phi/3}$  eigenvector of  $R_2$ , see (1.1). Therefore if  $\mathbf{z} \in C_1 \cap C_2$ , then we will get that

$$R_1 R_2(\mathbf{z}) = e^{-i\phi/3} R_1(\mathbf{z}) = e^{-2i\phi/3} \mathbf{z}$$

Hence  $\mathbf{z}$  is a  $\bar{u}^2 = e^{-2i\phi/3}$  eigenvector of  $R_1 R_2$ . Hence the sum of the other two eigenvalues of  $R_1 R_2$  is  $u(2 - |\rho|^2)$ . By the assumption  $|\rho| = 2 \cos \zeta$ , we know that  $R_1 R_2$  is not loxodromic, see Section 6.2 in [4]. Therefore each eigenvalue of  $R_1 R_2$  is of modulus one. Then we can get that the three eigenvalues of  $R_1 R_2$  will be  $\bar{u}^2$ ,  $-ue^{2i\zeta}$ ,  $-ue^{-2i\zeta}$  from the form of  $\text{tr}(R_1 R_2)$  in (2.3).  $\square$

**Remark 2.2.** We suppose that  $m \in \mathbb{N}$ ,  $m \geq 2$ . If  $|\rho| = 2 \cos(\pi/m)$  then  $br(R_1, R_2) = m$ , see Section 2.2 in [8] for details or more precisely [2, Proposition 2.3]. (In fact this is true if  $|\rho| = 2 \cos(k\pi/m)$  where  $k$  is coprime to  $m$ .) In the following Theorem 2.4, we suppose

$$|\rho| = |\sigma\tau - \bar{\rho}| = 2 \cos(\pi/m), \quad |\sigma| = |\tau| = 2 \cos(\pi/n),$$

which is the case of interest in [2].

If  $R_1, R_2$  are complex involutions ( $p = 2$ ), then the order of  $R_1 R_2$  will be of  $m$ .

Assume that

$$br(R_1, R_2) = br(R_1, R_3^{-1} R_2 R_3), \quad br(R_2, R_3) = br(R_1, R_3).$$

From Remark 2.2 and (2.3), our hypothesis on braiding implies that

$$|\rho| = |\sigma\tau - \bar{\rho}|, \quad |\sigma| = |\tau|.$$

Since we are free to choose the argument of two of the three parameters, we impose the condition that  $\sigma$  and  $\tau$  should be real and non-negative, which means that  $\text{Im}(\rho) = \text{Im}(\sigma\tau - \bar{\rho})$ . So the condition  $|\rho| = |\sigma\tau - \bar{\rho}|$  becomes either  $\sigma\tau = \rho + \bar{\rho}$  or  $\sigma\tau = 0$ . In the latter case the group is reducible, so we do not consider it. Hence we suppose  $\text{Re}(\rho) > 0$  and  $\sigma = \tau = \sqrt{\rho + \bar{\rho}}$ .

Suppose that  $|\rho| = 2 \cos(\pi/m)$  and  $\sigma = \tau = 2 \cos(\pi/n)$ , where  $m, n \in \mathbb{N}$  and  $m, n \geq 3$ . Then the matrices in (2.1) become:

$$R_1 = \begin{pmatrix} u^2 & \rho & -u\sqrt{\rho + \bar{\rho}} \\ 0 & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad (2.4)$$

$$R_2 = \begin{pmatrix} \bar{u} & 0 & 0 \\ -u\bar{\rho} & u^2 & \sqrt{\rho + \bar{\rho}} \\ 0 & 0 & \bar{u} \end{pmatrix}, \quad (2.5)$$

$$R_3 = \begin{pmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} & 0 \\ \sqrt{\rho + \bar{\rho}} & -u\sqrt{\rho + \bar{\rho}} & u^2 \end{pmatrix}. \quad (2.6)$$

Furthermore, the Hermitian form  $H$  (2.2) has signature  $(2, 1)$  if and only if

$$\begin{aligned} 0 &< \alpha|\rho|^2 + 2\alpha(\rho + \bar{\rho}) - \alpha^3 - i\bar{u}^{3/2}(\rho^2 + |\rho|^2) + iu^{3/2}(\bar{\rho}^2 + |\rho|^2) \\ &= 2\alpha(\rho + \bar{\rho}) - \alpha^3 - i\bar{u}^{3/2}\rho^2 + iu^{3/2}\bar{\rho}^2. \end{aligned} \quad (2.7)$$

**Proposition 2.3.** *Let*

$$S = \begin{pmatrix} \rho & u(1 - \rho - \bar{\rho}) & u^2\sqrt{\rho + \bar{\rho}} \\ \bar{u} & 0 & 0 \\ 0 & \bar{u}\sqrt{\rho + \bar{\rho}} & -1 \end{pmatrix}.$$

Then

$$\begin{aligned} (a) \quad &S^2 = R_1R_2R_3, \\ (b) \quad & \end{aligned}$$

$$\begin{aligned} SR_1S^{-1} &= R_1R_2R_1^{-1}, \\ SR_2S^{-1} &= R_1R_3R_1R_3^{-1}R_1^{-1}, \\ SR_3S^{-1} &= R_1R_3R_1^{-1}. \end{aligned}$$

In particular,

$$S(R_2R_3)S^{-1} = R_1R_3, \quad S(R_1R_3^{-1}R_2R_3)S^{-1} = R_1R_2.$$

Moreover,  $S$  is the only matrix in  $SU(2, 1)$  satisfying (a) and (b).

**Proof.** Suppose that  $S$  satisfies (b). The basis vectors  $\mathbf{v}_1 = [1, 0, 0]^t$ ,  $\mathbf{v}_2 = [0, 1, 0]^t$  and  $\mathbf{v}_3 = [0, 0, 1]^t$  are the polar vectors to the fixed complex geodesics of  $R_1, R_2, R_3$  respectively. Since  $SR_1S^{-1} = R_1R_2R_1^{-1}$ , we see that  $S$  sends  $\mathbf{v}_1$  to a vector that is polar to the fixed complex geodesic of  $R_1R_2R_1^{-1}$ , which is a non-zero multiple of  $R_1\mathbf{v}_2$ . Similarly for the other complex reflections. Therefore

$$S\mathbf{v}_1 = \lambda R_1\mathbf{v}_2, \quad S\mathbf{v}_2 = \mu R_1R_3\mathbf{v}_1, \quad S\mathbf{v}_3 = \nu R_1\mathbf{v}_3.$$

Hence any matrix  $S$  satisfying (b) has the form:

$$S = \begin{pmatrix} \lambda\rho & \mu u(1 - \rho - \bar{\rho}) & -\nu u\sqrt{\rho + \bar{\rho}} \\ \lambda\bar{u} & 0 & 0 \\ 0 & \mu\bar{u}\sqrt{\rho + \bar{\rho}} & \nu\bar{u} \end{pmatrix},$$

where  $\lambda, \mu, \nu \in \mathbb{C} - \{0\}$ . Now squaring  $S$  and comparing its entries with the entries of  $R_1R_2R_3$ , we see that if such a matrix  $S$  also satisfies (a), then we must have:

$$\lambda^2 = 1, \quad \lambda\mu = 1, \quad \mu\nu = -u, \quad \lambda\nu = -u, \quad \nu^2 = u^2.$$

Also, since  $S \in SU(2, 1)$  we have  $1 = \det(S) = -\lambda\mu\nu\bar{u}$ . The only solution to these equations is  $\lambda = \mu = 1$  and  $\nu = -u$ . Hence  $S$  has the form we claimed.

Finally, it is easy to check directly that the matrix  $S$  in the statement of the proposition lies in  $SU(2, 1)$  and satisfies (a) and (b).  $\square$

In the following we will classify all discrete triangle groups generated by  $R_1, R_2, R_3$  with the 2-fold symmetry given by  $S$  satisfying the conditions (a) and (b) in Proposition 2.3.

**Theorem 2.4.** *Let  $R_1, R_2, R_3$  be three complex reflections of order  $p$  (with  $p \geq 2$ ) in  $SU(2, 1)$  so that  $R_i$  keeps a complex geodesic  $C_i$  ( $i = 1, 2, 3$ ) invariant. Assume that there is  $S \in SU(2, 1)$  such that*

$$\begin{aligned} SR_1S^{-1} &= R_1R_2R_1^{-1}, \\ SR_2S^{-1} &= R_1R_3R_1R_3^{-1}R_1^{-1}, \\ SR_3S^{-1} &= R_1R_3R_1^{-1}, \\ S^2 &= R_1R_2R_3. \end{aligned}$$

Let  $\rho$  and  $\sigma$  be as in (2.3). Suppose  $|\rho| = 2 \cos(\pi/m)$  and  $|\sigma| = 2 \cos(\pi/n)$ , which implies that  $br(R_1, R_3) = n$ ,  $br(R_1, R_2) = m$  (where  $m, n \in \mathbb{N}$  and  $m, n \geq 3$ ). Suppose also that  $R_1R_2R_3$  is of finite order. Then the possible values for  $(n, m)$  are  $(3, 4)$ ,  $(3, 5)$ ,  $(4, 3)$ ,  $(5, 4)$ ,  $(8, 6)$  and  $(k, k)$  ( $k \in \mathbb{N}$  and  $k \geq 3$ ).

Moreover, in each case the group preserves a Hermitian form  $H$ . When  $(n, m)$  is one of  $(3, 5)$  or  $(k, k)$  for  $k \geq 5$  the form  $H$  has signature  $(2, 1)$  for all  $p \geq 2$ . For the other values of  $(n, m)$  the form  $H$  only has signature  $(2, 1)$  for the following values of  $p$ :

$$\begin{array}{lll} (3, 4), p \geq 5; & (4, 3), p \geq 4; & (5, 4), p \geq 3; \\ (8, 6), p \geq 3; & (3, 3), p \geq 4; & (4, 4), p \geq 3. \end{array}$$

Note that the solutions correspond to the following parameter values, or their complex conjugates:

| $(n, m)$ | $\rho$                                    | $s = \rho - 1$                                   | $\sigma = \tau$       |
|----------|---|--|-----------------------|
| $(3, 4)$ | $(1 + i\sqrt{7})/2$                       | $e^{2\pi i/7} + e^{4\pi i/7} + e^{-6\pi i/7}$    | 1                     |
| $(3, 5)$ | $2e^{2\pi i/5} \cos(\pi/5)$               | $e^{2\pi i/5} + e^{7\pi i/15} + e^{-13\pi i/15}$ | 1                     |
| $(4, 3)$ | 1   | 0  | $\sqrt{2}$            |
| $(5, 4)$ | $(1 + i\sqrt{3})(\sqrt{5} - i\sqrt{3})/4$ | $e^{-2\pi i/3} + e^{2\pi i/15} + e^{8\pi i/15}$  | $(1 + \sqrt{5})/2$    |
| $(8, 6)$ | $(1 + i)(1 - i/\sqrt{2})$                 | $e^{\pi i/2} + e^{\pi i/12} + e^{-7\pi i/12}$    | $\sqrt{2 + \sqrt{2}}$ |
| $(k, k)$ | $2e^{i\pi/k} \cos(\pi/k)$                 | $e^{2\pi i/k}$                                   | $2 \cos(\pi/k)$       |

## 3. THE PROOF

Firstly a direct computation will show that the symmetry  $S$  conjugates  $R_1R_2$  to  $R_1R_3^{-1}R_2R_3$  and conjugates  $R_2R_3$  to  $R_1R_3$ . It means that

$$br(R_1, R_2) = br(R_1, R_3^{-1}R_2R_3), \quad br(R_2, R_3) = br(R_1, R_3)$$

by recalling Remark 2.2 and (2.3). By the parameterization of the triangle groups in Section 2 and the assumption in Theorem 2.4, one could get the matrix representation of  $H, R_1, R_2, R_3$  as (2.2), (2.4), (2.5), (2.6), where

$$|\rho| = 2 \cos(\pi/m), \quad \sigma = \tau = 2 \cos(\pi/n).$$

Throughout the proof we let  $\zeta = \pi/m$  and  $\eta = \pi/n$ . Then by Proposition 2.3, we can get the unique matrix from of  $S \in \text{SU}(2, 1)$ .

Because of  $S^2 = R_1R_2R_3$ , we can restrict ourselves to  $S$ , which is elliptic of finite order. Equivalently, there exist  $a$  and  $b$  that are rational multiples of  $\pi$  for which:

$$\text{tr}(S) = -1 + \rho = e^{ia} + e^{ib} + e^{-i(a+b)}. \quad (3.1)$$

Observe that there is some ambiguity in the choice of  $a$  and  $b$ . First, we can permute the three terms in this expression, and so permute  $\{a, b, -a-b\}$ ; secondly we can change the sign of all three terms and, finally, since  $\text{tr}(S)$  is only defined up to multiplying by a cube root of unity, we can add the same integer multiple of  $2\pi/3$  to both  $a$  and  $b$ . We will use these operations to simplify things in our calculations below.

We denote  $\text{tr}(S)$  by  $s$ , then get that

$$|s|^2 = 1 + |\rho|^2 - 2 \text{Re}(\rho) = |e^{ia} + e^{ib} + e^{-i(a+b)}|^2, \quad (3.2)$$

$$\text{Re}(s) = -1 + \text{Re}(\rho) = \cos(a) + \cos(b) + \cos(a+b), \quad (3.3)$$

Recall that

$$\begin{aligned} |\rho|^2 &= 4 \cos^2 \zeta = 2 \cos(2\zeta) + 2, \\ \text{Re}(\rho) &= \frac{\sigma\tau}{2} = 2 \cos^2 \eta = \cos(2\eta) + 1. \end{aligned}$$

The above two equations can be simplified to

$$1 = \cos(2\zeta) - \cos(2\eta) - \cos(a-b) - \cos(a+2b) - \cos(2a+b), \quad (3.4)$$

$$0 = \cos(2\eta) - \cos(a) - \cos(b) - \cos(a+b). \quad (3.5)$$

In what follows we will repeatedly use the following result given by A. Monaghan, which generalizes the result of Conway and Jones for vanishing sums of cosines of rational multiples of  $\pi$ .

**Proposition 3.1.** [7, Theorem 2.4.3.1] *Suppose that we have at most five distinct rational numbers of  $\pi$ , for which some rational linear combination of*



their cosines is rational but no proper subset has this property. If  $\phi \in (0, \pi)$  and all other angles are normalized to lie in  $(0, \frac{\pi}{2})$ , then the appropriate linear combination is proportional to one of the following:

- (a)  $0 = \cos(\phi) + \cos(\phi + \frac{2\pi}{3}) + \cos(\phi + \frac{4\pi}{3})$ ,
- (b)  $0 = \cos(\phi) + \cos(\phi \pm \frac{2\pi}{5}) - \cos(\phi \pm \frac{2\pi}{15}) + \cos(\phi \pm \frac{7\pi}{15})$ ,
- (c)  $0 = \cos(\phi) - \cos(\phi \pm \frac{\pi}{5}) + \cos(\phi \pm \frac{\pi}{15}) - \cos(\phi \pm \frac{4\pi}{15})$ ,
- (d)  $\frac{1}{2} = \cos(\frac{\pi}{3})$ ,
- (e)  $\frac{1}{2} = \cos(\frac{\pi}{5}) - \cos(\frac{2\pi}{5})$ ,
- (f)  $\frac{1}{2} = \cos(\frac{\pi}{5}) - \cos(\frac{\pi}{15}) + \cos(\frac{4\pi}{15})$ ,
- (g)  $\frac{1}{2} = -\cos(\frac{2\pi}{5}) + \cos(\frac{2\pi}{15}) - \cos(\frac{7\pi}{15})$ ,
- (h)  $\frac{1}{2} = -\cos(\frac{\pi}{15}) + \cos(\frac{2\pi}{15}) + \cos(\frac{4\pi}{15}) - \cos(\frac{7\pi}{15})$ ,
- (i)  $\frac{1}{2} = \cos(\frac{\pi}{7}) - \cos(\frac{2\pi}{7}) + \cos(\frac{3\pi}{7})$ ,
- (j)  $\frac{1}{2} = \cos(\frac{\pi}{7}) - \cos(\frac{2\pi}{7}) + \cos(\frac{2\pi}{21}) - \cos(\frac{5\pi}{21})$ ,
- (k)  $\frac{1}{2} = \cos(\frac{\pi}{7}) + \cos(\frac{3\pi}{7}) - \cos(\frac{\pi}{21}) + \cos(\frac{8\pi}{21})$ ,
- (l)  $\frac{1}{2} = -\cos(\frac{2\pi}{7}) + \cos(\frac{3\pi}{7}) + \cos(\frac{4\pi}{21}) + \cos(\frac{10\pi}{21})$ ,
- (m)  $\frac{1}{2} = \cos(\frac{\pi}{7}) - \cos(\frac{\pi}{21}) + \cos(\frac{2\pi}{21}) - \cos(\frac{5\pi}{21}) + \cos(\frac{8\pi}{21})$ ,
- (n)  $\frac{1}{2} = -\cos(\frac{2\pi}{7}) + \cos(\frac{2\pi}{21}) + \cos(\frac{4\pi}{21}) - \cos(\frac{5\pi}{21}) + \cos(\frac{10\pi}{21})$ ,
- (o)  $\frac{1}{2} = \cos(\frac{3\pi}{7}) - \cos(\frac{\pi}{21}) + \cos(\frac{4\pi}{21}) + \cos(\frac{8\pi}{21}) + \cos(\frac{10\pi}{21})$ .

Since the right hand side of equation (3.4) is 1 (rather than 0 or 1/2), Monaghan's theorem implies that it must be a sum of (at least) two similar sums involving fewer cosines. We begin by showing that at least one of the cosines must itself be rational.

**Proposition 3.2.** *Suppose that  $\zeta = \pi/m$ ,  $\eta = \pi/n$  and  $a, b$  are rational multiples of  $\pi$  so that equations (3.4) and (3.5) hold. Then one of the cosines in equation (3.4) must be rational.*

**Proof.** Suppose that none of the cosines are rational. Then (3.4) splits into two rational sums, one of length two and the other of length three, neither of which has a rational subsum. By inspection from Proposition 3.1 we see that these two sums must have the value 0,  $\pm 1/2$ . Since they sum to 1, they must both be 1/2. Therefore, the sum of length 2 must be (e) and the sum of length 3 must be one of (f), (g) or (i).

- (1)  $1/2 = \cos(2\zeta) - \cos(2\eta) = -\cos(a-b) - \cos(a+2b) - \cos(2a+b)$ . Since  $\zeta = \pi/m$  and  $\eta = \pi/n$  the sum (e) implies  $2\zeta = \pi/5$  and  $2\eta = 2\pi/5$ .

For the second equation, there are certain symmetry operations on  $a$  and  $b$  described in the paragraph after equation (3.1) above. Up to these operations, we now list the possible values of  $a$  and  $b$ . In the first column we indicate which of the identities (a) to (o) in Proposition 3.1 we mainly used.

|     | $a - b$  | $a + 2b$   | $2a + b$   | $a$        | $b$       |
|-----|----------|------------|------------|------------|-----------|
| (f) | $\pi/15$ | $11\pi/15$ | $4\pi/5$   | $13\pi/45$ | $2\pi/9$  |
| (g) | $2\pi/5$ | $7\pi/15$  | $13\pi/15$ | $19\pi/45$ | $\pi/45$  |
| (i) | $2\pi/7$ | $4\pi/7$   | $6\pi/7$   | $-2\pi/7$  | $-4\pi/7$ |

Using  $2\eta = 2\pi/5$ , we see that none of the values in this table satisfy (3.5). Therefore we get no solutions.

- (2)  $1/2 = \cos(2\zeta) - \cos(a - b) = -\cos(2\eta) - \cos(a + 2b) - \cos(2a + b)$ . The first equation gives  $2\zeta = \pi/5$  as in case (1) and so  $a - b = 2\pi/5$ . Since  $a - b = (2a + b) - (a + 2b)$  the difference of two of the angles in the second equation must be  $2\pi/5$ . By inspection, we see the only solution is  $a + 2b = 7\pi/15$  and  $2a + b = 13\pi/15$ . This means  $2\eta = 2\pi/5$  and we are back in case (1).
- (3)  $1/2 = -\cos(2\eta) - \cos(a - b) = \cos(2\zeta) - \cos(a + 2b) - \cos(2a + b)$ . The first equation gives  $2\eta = 2\pi/5$  as in (1) and so  $a - b = 4\pi/5$ . Substituting in the second equation, we see  $a + 2b = -\pi/15$  and  $2a + b = 11\pi/15$ . Thus  $2\zeta = \pi/5$  and we are back in case (1) again.
- (4)  $1/2 = -\cos(a - b) - \cos(a + 2b) = \cos(2\zeta) - \cos(2\eta) - \cos(2a + b)$ . Up to symmetries of  $a$ ,  $b$  and  $-a - b$ , the first sum implies that  $a - b = 2\pi/5$  and  $a + 2b = 4\pi/5$ . Hence  $2a + b = 6\pi/5$  and so the second sum must be (f). Thus  $\cos(2\zeta) = \cos(2\pi/m) = \cos(4\pi/15)$  or  $\cos(2\eta) = \cos(2\pi/n) = -\cos(4\pi/15)$ , so either  $m$  or  $n$  is not an integer. Therefore there are no solutions.  $\square$

As a consequence of this result, we can consider separate cases where each of the cosines in (3.4) is rational. If either  $\cos(2\zeta)$  or  $\cos(2\eta)$  is rational it must be 0 or  $\pm 1/2$  since  $\zeta = \pi/m$  and  $\eta = \pi/n$  where  $m$  and  $n$  are at least 3. If one of the other three cosines is rational we can use the allowable symmetries of  $a$  and  $b$ , we to assume that  $\cos(a - b)$  is rational. We treat each of these cases separately below. First we eliminate a simple situation which gives us many solutions and will recur in the different cases.

**Lemma 3.3.** *Suppose that  $\cos(2\zeta) = \cos(2\eta)$ , or equivalently  $m = n$ , then putting  $s = e^{\pm 2\pi i/m}$  gives a solution to equations (3.4) and (3.5) for all  $m \geq 3$ .*

**Proof.** Substituting  $\cos(2\zeta) = \cos(2\eta)$  into (3.4) gives:

$$0 = 1 + \cos(a - b) + \cos(a + 2b) + \cos(2a + b)$$

$$\begin{aligned}
&= 2 \cos^2((a-b)/2) + 2 \cos((a-b)/2) \cos(3(a+b)/2) \\
&= 4 \cos((a-b)/2) \cos((a+2b)/2) \cos((2a+b)/2).
\end{aligned}$$

Therefore one of  $(a-b)$ ,  $(a+2b)$  or  $(2a+b)$  is an odd multiple of  $\pi$ . Without loss of generality, we suppose that  $a+2b = (2k+1)\pi$ . Then we get  $-a-b = b - (2k+1)\pi$  which yields  $s = e^{ia}$ , where  $a$  is a rational multiple of  $\pi$ . Because  $\operatorname{Re}(s) = -1 + \operatorname{Re}(\rho) = -1 + \frac{|\sigma|^2}{2} = \cos(2\eta)$ , we see that  $\cos(a) = \cos(2\pi/m) = \cos(2\pi/n)$ .

Now we consider the signature of the Hermitian form

$$\operatorname{Det}(H) = ie^{-\frac{4\theta+3\phi}{2}i}(-1 + e^{(2\theta+\phi)i})(e^{i\theta} + e^{i\phi})^2.$$

TABLE 3.1. Signature of Hermitian form

| $s$   | $(2, 1)$   | degenerate | $(3, 0)$ |
|---|------------|------------|----------|
| $e^{\frac{2\pi i}{3}}$ ( $m = n = 3$ )        | $p \geq 4$ | $p = 3$    | $p = 2$  |
| $e^{\frac{2\pi i}{4}}$ ( $m = n = 4$ )        | $p \geq 3$ | $p = 2$    | none     |
| $e^{\frac{2\pi i}{k}}$ ( $m = n = k \geq 5$ ) | $p \geq 2$ | none       | none     |

In this case, we get the solution  $n = m$ . □

We now consider the cases where  $\cos(2\zeta)$ ,  $\cos(2\eta)$  or  $\cos(a-b)$  are rational. We will use the following result proved by Parker when he was analyzing the triangle groups with 3-fold symmetry [10]. In [10] the last two cases were missed out, but this was corrected in [2].

**Proposition 3.4.** [10, Proposition 3.2] *Let  $\theta$ ,  $a$  and  $b$  be rational multiples of  $\pi$ . Write  $s = e^{ia} + e^{ib} + e^{-i(a+b)}$ . Then the only possible solutions to the equation*

$$\cos(2\theta) - \cos(a-b) - \cos(a+2b) - \cos(2a+b) = \frac{1}{2}$$

give rise to the following values of  $\theta$  and  $s$ , up to changing the sign of  $\theta$  and up to conjugating  $s$  and multiplying it by a power of  $\omega = e^{2\pi i/3}$ :

- (i)  $2\theta = 2\pi/3$  and  $s = -e^{-i\psi/3}$  for some angle  $\psi$  that is a rational multiple of  $\pi$ ;
- (ii)  $2\theta = \psi$  and  $s = e^{2i\psi/3} + e^{-i\psi/3} = e^{i\psi/6} 2 \cos \frac{\psi}{2}$  for some angle  $\psi$  that is a rational multiple of  $\pi$ ;
- (iii)  $2\theta = \pi/3$  and  $s = e^{i\pi/3} + e^{-i\pi/6} 2 \cos \frac{\pi}{4}$ ;
- (iv)  $2\theta = \pi/5$  and  $s = e^{i\pi/3} + e^{-i\pi/6} 2 \cos \frac{\pi}{5}$ ;

- (v)  $2\theta = 3\pi/5$  and  $s = e^{i\pi/3} + e^{-i\pi/6}2 \cos \frac{2\pi}{5}$ ;
- (vi)  $2\theta = \pi/2$  and  $s = e^{2\pi i/7} + e^{4\pi i/7} + e^{-6\pi i/7}$ ;
- (vii)  $2\theta = \pi/2$  and  $s = e^{2\pi i/9} + e^{-i\pi/9}2 \cos \frac{2\pi}{5}$ ;
- (viii)  $2\theta = \pi/2$  and  $s = e^{2\pi i/9} + e^{-i\pi/9}2 \cos \frac{4\pi}{5}$ ;
- (ix)  $2\theta = \pi/7$  and  $s = e^{2\pi i/9} + e^{-i\pi/9}2 \cos \frac{2\pi}{7}$ ;
- (x)  $2\theta = 5\pi/7$  and  $s = e^{2\pi i/9} + e^{-i\pi/9}2 \cos \frac{4\pi}{7}$ ;
- (xi)  $2\theta = 3\pi/7$  and  $s = e^{2\pi i/9} + e^{-i\pi/9}2 \cos \frac{6\pi}{7}$ ;
- (xii)  $2\theta = 2\pi/5$  and  $s = 1 + 2 \cos \frac{2\pi}{5}$ ;
- (xiii)  $2\theta = 4\pi/5$  and  $s = 1 + 2 \cos \frac{4\pi}{5}$ .

Note that for the groups Parker was considering  $s = e^{ia} + e^{ib} + e^{-ia-ib}$  was the trace of  $R_1J$ , whereas in our case it is the trace of  $S$ . In the cases where  $\cos(2\zeta) = 1/2$  or  $\cos(2\eta) = -1/2$  then equation (3.4) reduces to the equation from Proposition 3.4, and we can use that result to find solutions.

**Lemma 3.5.** *Suppose that  $\cos(2\eta)$  is rational. Then the only solutions to (3.4) and (3.5) are  $\cos(2\zeta) = \cos(2\pi/m)$  and  $\cos(2\eta) = \cos(2\pi/n)$  where  $(n, m)$  is one of  $(3, 3)$ ,  $(3, 4)$ ,  $(3, 5)$ ,  $(4, 3)$ ,  $(4, 4)$  or  $(6, 6)$ .*

**Proof.** Since  $\cos(2\eta)$  is rational and not equal to  $\pm 1$  it can only be 0 or  $\pm 1/2$ . We treat each case separately.

(1)  $\cos(2\eta) = -\frac{1}{2}$ , which gives  $n = 3$ . Note that

$$\frac{1}{2} = \cos(2\eta) + 1 = \operatorname{Re}(\rho) = \operatorname{Re}(s) + 1$$

and so  $\operatorname{Re}(s) = -1/2$ . We rewrite (3.4) to give the equation from Proposition 3.4 with  $\theta = \zeta$ .

By direct calculation, we just need to consider cases (i), (ii) and (vi) because of  $\operatorname{Re}(s) = -1/2$ .

(i)  $s = -e^{-i\psi/3}$  and so  $|s| = 1$ . This yields that  $|\rho| = 2 \cos(\pi/m) = 1$ , and so  $m = 3$ . By considering  $\operatorname{Re}(s) = -\cos(\theta/3)$ , we know that  $\theta = \pm\pi + 6k\pi$  ( $k \in \mathbb{Z}$ ) which means that  $s = -e^{\mp i\pi/3}$ . From (2.7), we get that

$$\operatorname{Det}(H) = \mp\sqrt{3} \cos(\phi/2) + \sin(\phi/2) - 2 \sin(3\phi/2).$$

We list the corresponding signature of Hermitian form for different  $s$  in Table 3.2. In this case, we get that  $n = m = 3$ .

TABLE 3.2. Signature of Hermitian form

| $s$            | (2, 1)     | degenerate | (3, 0)     |
|----------------|------------|------------|------------|
| $-e^{-i\pi/3}$ | $p \geq 4$ | $p = 3$    | $p = 2$    |
| $-e^{i\pi/3}$  | none       | $p = 6$    | $p \neq 6$ |

(ii)  $s = e^{2i\psi/3} + e^{-i\psi/3} = e^{i\psi/6} 2 \cos(\psi/2)$  where  $\psi = 2\theta$ . By solving

$$\begin{aligned} -1/2 &= \operatorname{Re}(s) = \cos(4\theta/3) + \cos(2\theta/3) \\ &= 2 \cos^2(2\theta/3) + \cos(2\theta/3) - 1 \end{aligned}$$

we obtain  $\cos(2\theta/3) = (-1 \pm \sqrt{5})/4$ . That is,  $2\theta/3 = \pm 2\pi/5 + 2k\pi$  or  $\pm 4\pi/5 + 2k\pi$ . Hence  $\theta = \pm 3\pi/5 + 3k\pi$  or  $\pm 6\pi/5 + 3k\pi$ . The only solution to  $|s| = 2|\cos(\theta)| = 2\cos(\pi/m)$  is  $m = 5$  (coming from  $2\theta/3 = 4\pi/5 - 2\pi$ ).

Therefore  $s = e^{2\pi i/5} + e^{4\pi i/5}$  or  $s = e^{-2\pi i/5} + e^{-4\pi i/5}$ . In these cases we find, respectively, that:

$$\begin{aligned} \operatorname{Det}(H) &= -\sqrt{5 + 2\sqrt{5}} \cos \frac{\phi}{2} - (2 + \sqrt{5} + 4 \cos \phi) \sin \frac{\phi}{2}, \\ \operatorname{Det}(H) &= \sqrt{5 + 2\sqrt{5}} \cos \frac{\phi}{2} - (2 + \sqrt{5} + 4 \cos \phi) \sin \frac{\phi}{2}. \end{aligned}$$

TABLE 3.3. Signature of Hermitian form

| $s$   | (2, 1)     | degenerate | (3, 0)     |
|---|------------|------------|------------|
| $e^{-\frac{2\pi i}{5}} + e^{-\frac{4\pi i}{5}}$ | $p \leq 7$ | none       | $p \geq 8$ |
| $e^{\frac{2\pi i}{5}} + e^{\frac{4\pi i}{5}}$   | $p \geq 2$ | none       | none       |

In this case, we get that  $n = 3$ ,  $m = 5$ .

(vi) Since  $s = e^{2\pi i/7} + e^{4\pi i/7} + e^{-6\pi i/7} = (-1 + \sqrt{7}i)/2$ , it follows that  $\operatorname{Re}(s) = -1/2$  and  $|s| = \sqrt{2}$  which indicates that  $m = 4$ . A simple calculation yields that

$$\operatorname{Det}(H) = \frac{1}{2}(1 - 8 \cos \phi) \sin \frac{\phi}{2}$$

from which it follows that the signature of the Hermitian form will be of (2, 1) for  $p \geq 5$ , otherwise it will be positive. In this case, we get that  $n = 3$ ,  $m = 4$ .

Therefore we obtain the solutions  $(n, m) = (3, 3)$ ,  $(3, 4)$  and  $(3, 5)$ .

- (2)  $\cos(2\eta) = 0$ . Now we have  $|\sigma|^2 = 2$  which yields  $\operatorname{Re}(s) = 0$ . Therefore one can get the following two equations

$$\begin{cases} \cos(2\zeta) - \cos(a - b) - \cos(a + 2b) - \cos(2a + b) = 1, \\ \cos a + \cos b + \cos(a + b) = 0. \end{cases} \quad (3.6)$$

Since the first of these has 1 on the right hand side, it must split as the sum of (at least) two minimal subsums. Treating these case by case we see that the only possibilities are  $\cos(2\zeta) = 0$ , which yields  $m = n = 4$  and  $\cos(2\zeta) = -1/2$ , which gives  $n = 4$  and  $m = 3$ . The former case is a particular instance of Lemma 3.3. In the latter case we rewrite (3.2) as

$$|s|^2 = 1 + |\rho|^2 - 2 \operatorname{Re}(\rho) = 1 + 2 \cos(2\zeta) - 2 \cos(2\eta) = 0.$$

Therefore, the only solution is  $s = 0$ , or equivalently  $\rho = 1$ . This implies that

$$\operatorname{Det}(H) = -2 \sin \frac{3\phi}{2} = -2 \sin \frac{3\pi}{p},$$

and the signature of the Hermitian form will be positive if  $p = 2$ , degenerate if  $p = 3$ , negative (of signature  $(2, 1)$ ) if  $p \geq 4$ . Therefore in this case we get that  $n = 4$ ,  $m = 3$ . Hence the only solutions we get in this case are  $(n, m) = (4, 3)$  and  $(4, 4)$ .

- (3)  $\cos(2\eta) = 1/2$ . Now we have  $|\sigma|^2 = 3$  from which it follows that  $\operatorname{Re}(s) = 1/2$ . We rewrite the two equations

$$\begin{cases} \cos(2\zeta) - \cos(a - b) - \cos(a + 2b) - \cos(2a + b) = \frac{3}{2}, \\ \cos a + \cos b + \cos(a + b) = \frac{1}{2}. \end{cases} \quad (3.7)$$

If the second equation is irreducible, then it must be one of Proposition 3.1 parts (f), (g) or enum:posi:1:i. We see in each case that the angles involved do not sum to 0 (making each cosine positive, the sum is  $\pi$  times the ratio of two odd integers for each choice of sign). If the second equation splits as the sum of two rational subsums then, without loss of generality,  $\cos(a)$  is rational. Hence it is in the set  $\{0, \pm 1/2, \pm 1\}$ . Simple trigonometry shows that

$$\begin{aligned} 2 \cos(a/2) \cos(a/2 + b) &= \cos(b) + \cos(a + b) \\ &= 1/2 - \cos(a), \\ \cos(a + 2b) + 1 &= 2 \cos^2(a/2 + b) \\ &= (1/2 - \cos(a))^2 / (1 + \cos(a)), \\ \cos(a - b) + \cos(2a + b) &= 2 \cos(3a/2) \cos(a/2 + b) \end{aligned}$$

$$= -2(1/2 - \cos(a))^2.$$

Substituting these identities in the first equation, we see that  $\cos(2\zeta)$  is a rational function of  $\cos(a)$ , and so is rational. Substituting the different values of  $\cos(a)$  gives a solution with  $\zeta = \pi/m$  only when  $\cos(a) = \pm 1/2$ . In both cases,  $\cos(2\zeta) = 1/2$  and so  $m = 6$ . Thus we obtain the solution  $(n, m) = (6, 6)$ .  $\square$

**Lemma 3.6.** *Suppose that  $\cos(2\zeta)$  is rational. Then the only solutions to (3.4) and (3.5) are  $\cos(2\zeta) = \cos(2\pi/m)$  and  $\cos(2\eta) = \cos(2\pi/n)$ , where  $(n, m)$  is one of  $(3, 3)$ ,  $(4, 3)$ ,  $(4, 4)$ ,  $(5, 4)$ ,  $(6, 6)$  or  $(8, 6)$ .*

**Proof.** Since  $\cos(2\zeta)$  is rational and not equal to  $\pm 1$  it can only be 0 or  $\pm 1/2$ . We treat each case separately.

(1)  $\cos(2\zeta) = 1/2$ , which gives  $m = 6$ . In this case, we know

$$|s|^2 = 1 + 2\cos(2\zeta) - 2\cos(2\eta) = 2 - 2\cos(2\eta).$$

In this case, we rewrite equation (3.4) to give the equation from Proposition 3.4 with  $2\theta = \pi - 2\eta$ . Checking one by one, we will find that there is no value of  $s$  in Proposition 3.4 satisfying (3.5) except the cases (i) and (ii). For (i) we have  $2\eta = \pi - 2\theta = \pi/3$  and so  $n = 6$  (we have analyzed this case previously). For (ii), we have  $\psi = \pi - 2\eta$  and  $s = e^{2i\pi/3 - 4i\eta/3} + e^{-i\pi/3 + 2i\eta/3}$ . Substituting in equation (3.5) gives

$$\begin{aligned} 0 &= \cos(2\eta) - \operatorname{Re}(s) \\ &= -\cos(\pi - 2\eta) - \cos(2\pi/3 - 4\eta/3) - \cos(\pi/3 - 2\eta/3) \\ &= -\cos(2\pi/3 - 4\eta/3)(1 + 2\cos(\pi/3 - 2\eta/3)). \end{aligned}$$

The only solution with  $\eta = \pi/n$  is when  $2\pi/3 - 4\eta/3 = \pi/2$ . That is,  $n = 8$ . By calculating  $\operatorname{Det}(H) = -2\cos(\phi)(1 + 2\sin\phi)$ , we see that  $H$  is of signature  $(3, 0)$  for  $p = 2$  and is of signature  $(2, 1)$  for any  $p \geq 3$ . In this case, we get  $(n, m) = (6, 6)$  or  $(8, 6)$ .

(2)  $\cos(2\zeta) = 0$ , which gives  $m = 4$ . Then we get that  $|\rho|^2 = 2$  and  $|s|^2 = 3 - |\sigma|^2$ . Also (3.4) can be replaced by

$$-\cos(2\eta) - \cos(a - b) - \cos(a + 2b) - \cos(2a + b) = 1.$$

We have already analyzed the case where  $\cos(2\eta) = 0$  or  $-1/2$ , which lead to the solution  $(n, m) = (3, 4)$  or  $(4, 4)$ . If  $\cos(2\eta) = 1/2$ , then  $|\sigma|^2 = 3$  induces  $s = 0$ , which contradicts  $\operatorname{Re}(s) = -1 + |\sigma|^2/2 = 1/2$ . Then it suffices for us to consider the following possible values due to  $\eta = \pi/n$ ,

|     | $2\eta$  | $a - b$  | $a + 2b$  | $2a + b$   | $a$      | $b$       |
|-----|----------|----------|-----------|------------|----------|-----------|
| (g) | $2\pi/5$ | $2\pi/3$ | $7\pi/15$ | $17\pi/15$ | $3\pi/5$ | $-\pi/15$ |
| (e) | $2\pi/5$ | $2\pi/5$ | $4\pi/5$  | $6\pi/5$   | $8\pi/5$ | $2\pi/15$ |

From this table, we know that  $n = 5$  and the pair values  $a = 3\pi/5$  and  $b = -\pi/15$  do not satisfy the equation (3.5). However the second line  $a = 8\pi/5$  and  $b = 2\pi/15$  satisfy the equation (3.5) by applying the equation (g) in Proposition 3.1. Then we calculate the signature of the Hermitian form  $H$  using (2.7). We see that  $H$  is of signature  $(3, 0)$  for  $p = 2$  and is of signature  $(2, 1)$  for any  $p \geq 3$ . In this case, we get that  $m = 4$  and  $n = 5$ .

(3)  $\cos(2\zeta) = -\frac{1}{2}$ . It follows that  $m = 3$  and

$$-\cos(2\eta) - \cos(a - b) - \cos(a + 2b) - \cos(2a + b) = \frac{3}{2}.$$

Also,  $\cos(2\zeta) = -1/2$  implies  $|\rho| = 1$  and so  $\cos(2\eta) + 1 = \operatorname{Re}(\rho) \leq 1$ . This means that  $\cos(2\eta) \leq 0$  and so either  $\cos(2\eta) = \cos(2\pi/n) = -1/2$  or 0. We have analyzed both of these cases already. These give solutions  $(n, m) = (3, 3)$  or  $(4, 3)$ .  $\square$

Now we begin to consider the remaining case in which  $\cos(a - b)$  is rational.

**Lemma 3.7.** *Suppose that  $\cos(a - b) = -1$ , then  $\cos(2\zeta) - \cos(2\eta) = 0$ , and the possible solutions are given in Lemma 3.3 in which  $n = m$ .*

**Proof.** It follows from  $\cos(a - b) = -1$  that  $b = a + (2k + 1)\pi$ . Hence we have  $\cos(a + 2b) = \cos(3a)$  and  $\cos(2a + b) = -\cos(3a)$ . Therefore, equation (3.4) reduces to  $\cos(2\zeta) - \cos(2\eta) = 0$ , which we have already treated in Lemma 3.3.  $\square$

**Lemma 3.8.** *Suppose that  $\cos(a - b) = -1/2$ ,  $\cos(2\zeta)$  and  $\cos(2\eta)$  are not rational,  $\cos(2\zeta) - \cos(2\eta) \neq 0$ . Then we get no solutions for  $n, m$  such that (3.4) and (3.5) hold.*

**Proof.** It follows from  $\cos(a - b) = -1/2$  that  $b = a \pm 2\pi/3 + 2k\pi$ . Hence we have  $\cos(a + 2b) = \cos(3a \mp 2\pi/3)$  and  $\cos(2a + b) = \cos(3a \pm 2\pi/3)$ . Therefore equation (3.4) becomes

$$\begin{aligned} 1/2 &= 1 + \cos(a - b) \\ &= \cos(2\zeta) - \cos(2\eta) - \cos(a + 2b) - \cos(2a + b) \\ &= \cos(2\zeta) - \cos(2\eta) + \cos(3a). \end{aligned}$$

Since we have supposed that  $\cos(2\zeta)$  and  $\cos(2\eta)$  are not rational, the only way this equation can split into to rational subsums is for  $\cos(3a)$  to be



rational. Investigating the different possibilities, we see that (3.5) then implies  $\cos(2\eta)$  is rational.

Now suppose the equation does not split into two rational sums of cosines. We list the possible values of  $2\zeta$ ,  $2\eta$ ,  $a$ ,  $b$  in Table 3.4, up to the allowable symmetries of  $a$  and  $b$ .

However, we note that there are no values of  $2\eta$ ,  $a$ ,  $b$  in the list satisfying (3.5). Therefore there are no solutions for  $n, m$ .  $\square$

TABLE 3.4.

|     | $2\zeta$  | $2\eta$  | $3a$      | $a - b$  | $a$       | $b$         |
|-----|-----------|----------|-----------|----------|-----------|-------------|
| (e) | $\pi/5$   | $2\pi/5$ | $\pi/2$   | $2\pi/3$ | $\pi/6$   | $-\pi/2$    |
| (e) | $2\pi/5$  | $\pi/5$  | 0         | $2\pi/3$ | 0         | $-2\pi/3$   |
| (f) | $\pi/5$   | $\pi/15$ | $4\pi/15$ | $2\pi/3$ | $4\pi/45$ | $-26\pi/45$ |
| (g) | $2\pi/15$ | $2\pi/5$ | $8\pi/15$ | $2\pi/3$ | $8\pi/45$ | $-22\pi/45$ |
| (i) | $\pi/7$   | $2\pi/7$ | $3\pi/7$  | $2\pi/3$ | $\pi/7$   | $-11\pi/21$ |

**Lemma 3.9.** *Suppose that  $\cos(a - b) = 0, 1/2$  or  $1$ ,  $\cos(2\zeta)$  and  $\cos(2\eta)$  are not rational,  $\cos(2\zeta) - \cos(2\eta) \neq 0$ . Then there are no solutions for  $n, m$  satisfying both (3.4) and (3.5).*

**Proof.** We immediately get that

$$\cos(2\zeta) - \cos(2\eta) - \cos(a + 2b) - \cos(2a + b) = 1 \text{ or } \frac{3}{2} \text{ or } 2. \quad (3.8)$$

Since the right hand side is not  $0, \pm 1/2$ , we see that this sum must split into shorter rational sums of cosines. We break down into the following three cases.

(1) Case  $\cos(2\zeta) - \cos(2\eta) = \pm 1/2$ .

(i) Suppose  $\cos(2\zeta) - \cos(2\eta) = 1/2$ . Note that  $\zeta = \pi/m$  and  $\eta = \pi/n$ , where  $m, n \in \mathbb{N}$ . Therefore we know that  $(n, m)$  is  $(5, 10)$  and

$$\cos(a - b) + \cos(a + 2b) + \cos(2a + b) = -\frac{1}{2}.$$

We have supposed that  $\cos(a - b)$  is rational, then using elementary trigonometry arguments, we see that

$$2 \cos((a - b)/2) \cos(3(a + b)/2) = -\frac{1}{2} - \cos(a - b).$$

Squaring both sides and rearranging gives

$$\cos(3a + 3b) = \frac{\cos^2(a - b) - 3/4}{\cos(a - b) + 1}.$$

We have assumed that either  $\cos(a - b) = 0$  or  $\cos(a - b) = 1/2$  or  $\cos(a - b) = 1$ , which means that  $\cos(3a + 3b) = -3/4$  or  $-1/3$  or  $-1/8$ . It gives a contradiction here.

(ii)  $\cos(2\zeta) - \cos(2\eta) = -1/2$ . It follows that

$$\cos(a + 2b) - \cos(2a + b) = -\frac{3}{2} \text{ or } -2 \text{ or } -\frac{5}{2}.$$

This sum must again split and so both cosines are rational. Therefore the possible values for  $\cos(a + 2b)$  are just  $-1$  or  $-1/2$  which are equivalent to the case where  $\cos(a - b)$  is this value, see Lemma 3.7 and Lemma 3.8. However we assumed  $\cos(2\zeta) - \cos(2\eta) \neq 0$ , therefore there are no solutions for  $n, m$  satisfying both (3.4) and (3.5).

(2) Assume that  $\cos(2\zeta) - \cos(x)$  (or  $\cos(2\eta) + \cos(y)$ ) is  $1/2$  or  $-1/2$ , where  $x, y \in \{a + 2b, 2a + b\}$ .

Recalling the equation (3.8),  $\cos(2\zeta) - \cos(x) = \pm 1/2$  means that  $\cos(2\eta) + \cos(y)$  is one of the values  $\{-5/2, -2, -3/2, -1, -1/2\}$ . We just need to consider the case  $\cos(2\eta) + \cos(y) = -1/2$ , because other values of  $\cos(2\eta) + \cos(y)$  mean that  $\cos(2\eta)$  will be rational. Without loss of generality, we suppose that  $x = a + 2b, y = 2a + b$  and list the values of  $2\zeta, a + 2b, 2\eta, 2a + b$  and corresponding  $a - b$ :

|     | $2\zeta$ | $a + 2b$  | $2\eta$  | $2a + b$ | $a - b$  |
|-----|----------|-----------|----------|----------|----------|
| (e) | $\pi/5$  | $2\pi/5$  | $2\pi/5$ | $4\pi/5$ | $2\pi/5$ |
| (e) | $\pi/5$  | $-2\pi/5$ | $2\pi/5$ | $4\pi/5$ | $6\pi/5$ |

There are no values of  $a - b$  such that  $\cos(a - b) = 0$  or  $\cos(a - b) = 1/2$  or  $\cos(a - b) = 1$ . Therefore there are no solutions for  $n, m$  satisfying both (3.4) and (3.5) in this case.

(3) Suppose that  $\cos(x)$  is rational, where  $x \in \{a + 2b, 2a + b\}$ . By suitable changes of  $a$  and  $b$ , the cases  $\cos(a + 2b)$  or  $\cos(2a + b)$  is  $-1/2$  or  $-1$  are equivalent to the cases in Lemma 3.7 and Lemma 3.8. Therefore there are no solutions for  $n, m$  because we supposed  $\cos(2\zeta) - \cos(2\eta) \neq 0$ .

Then we consider the condition for  $\cos(x)$  to be  $0, 1/2$  or  $1$  and suppose that  $x = a + 2b$ . We get that

$$\cos(2\zeta) - \cos(2\eta) - \cos(2a + b) \in \left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\},$$

which can be reduced to  $\cos(2\zeta) - \cos(2a + b)$  or  $\cos(2\eta) + \cos(2a + b)$  is rational which has been considered above.

Now we can get that there are no solutions for  $n, m$  satisfying both (3.4) and (3.5) under the conditions in Lemma 3.9.  $\square$

We sum up all the possible values for  $n, m$  from above process,

Lemma 3.3  $n = m \geq 3$ ;

Lemma 3.5  $(n, m) \in (3, 3), (3, 4), (3, 5), (4, 3), (4, 4)$  or  $(6, 6)$ ;

Lemma 3.6  $(n, m) \in (3, 3), (4, 3), (4, 4), (5, 4), (6, 6)$  or  $(8, 6)$ ;

which we desired. Also we could see the range of  $p$  for each possible value to hold from the above analysis.

**Remark 3.10.** Note that the new candidates for  $(n, m)$  to be  $(5, 4), (4, 3)$  and  $(8, 6)$  do not appear on Thompson's list in [14]. However referring to [2], in what follows we will see that the triangle groups for  $(n, m)$  to be  $(5, 4)$  corresponds to Thompson groups  $\mathbf{S}_2$  and the triangle groups for  $(m, n)$  to be  $(4, 3)$  is of actually Mostow groups with braiding  $(2, 3, 4; 4)$ . The pair  $(n, m) = (8, 6)$  was also found by Deraux when he was making a similar computer search to Thompson (private communication).

Case 1:  $(n, m) = (5, 4)$ . Suppose that  $M_1, M_2, M_3$  are three complex reflections of order  $p$ , which satisfy

$$\begin{aligned} br(M_1, M_2) &= 4, \\ br(M_1, M_3) &= br(M_2, M_3) = 3, \\ br(M_1, M_3^{-1}M_2M_3) &= 5. \end{aligned}$$

Actually,  $M_1, M_2, M_3$  will be Thompson group  $\mathbf{S}_2$ . Write  $R_1 = M_2^{-1}M_1M_2$ ,  $R_2 = M_1M_2M_1^{-1}$ ,  $R_3 = M_3$ . We claim that

$$br(R_1, R_2) = br(R_1, R_3^{-1}R_2R_3) = 4, \quad br(R_1, R_3) = br(R_2, R_3) = 5.$$

First, observe, we also have  $br(M_2^{-1}M_1M_2, M_3) = br(M_1^{-1}M_2M_1, M_3) = 5$ . Thus

$$\begin{aligned} br(R_1, R_3) &= br(M_2^{-1}M_1M_2, M_3) = 5, \\ br(R_2, R_3) &= br(M_1M_2M_1^{-1}, M_3) = 5. \end{aligned}$$

Using  $br(M_1, M_2) = 4$ , we have

$$R_1R_2 = (M_2^{-1}M_1M_2)(M_1M_2M_1^{-1}) = M_2^{-1}(M_2M_1M_2M_1)M_1^{-1} = M_1M_2.$$

Hence  $br(R_1, R_2) = br(M_1, M_2) = 4$ . We denote  $M_1, M_1^{-1}$  by  $1, \bar{1}$  simply and so on. Now we consider

$$\begin{aligned} R_1R_3^{-1}R_2R_3 &= M_2^{-1}M_1M_2M_3^{-1}M_1M_2M_1^{-1}M_3 \\ &= \bar{2}12\bar{3}12\bar{1}3 \\ &= (123123)(\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \cdot \bar{2}12\bar{3}12\bar{1}3 \cdot 123123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \\ &= (123123)(\bar{3}\bar{2}\bar{1}\bar{3} \cdot 12\bar{1} \cdot \bar{3}12\bar{1}3 \cdot 123123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \\ &= (123123)(\bar{3}\bar{2}(\bar{1}\bar{3}1)2(\bar{1}\bar{3}1)2(\bar{1}\bar{3}1)23123)\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}\bar{1} \end{aligned}$$

$$\begin{aligned}
&= (123123)(\overline{31}31\overline{21}\overline{31}123)\overline{32}\overline{13}\overline{21} \\
&= (123123)(1\overline{3}\overline{2}3\overline{2}3)\overline{32}\overline{13}\overline{21} \\
&= (123123)(12)\overline{32}\overline{13}\overline{21}.
\end{aligned}$$

Since  $R_1R_3^{-1}R_2R_3$  is conjugate to  $M_1M_2$  we see that

$$br(R_1, R_3^{-1}R_2R_3) = br(M_1, M_2) = 4$$

as claimed. In particular, this shows that this case is equivalent to Thompson groups  $\mathbf{S}_2$ .

Case 2:  $(n, m) = (4, 3)$ . In this case, it is easy to check that  $R_2, R_3$  (also  $R_3, R_1$ ) braid with length 4,  $R_1, R_2$  (also  $R_1, R_3^{-1}R_2R_3$ ) braid with length 3,  $R_1, R_2R_3R_2^{-1}$  (also  $R_3, R_1R_2R_1^{-1}$ ) braid with length 2 (i.e. they commute) and  $R_1R_2R_3$  is regular elliptic of order 3. Note that  $\text{Det}(H) < 0$  when  $p \geq 4$ .

As the same fashion in [5], we define  $\iota$  by the reflection of group that acts on the generating set  $(R_1, R_2, R_3)$  as follows,

$$\iota(R_1) = R_1, \quad \iota(R_2) = R_1R_2R_1^{-1}, \quad \iota(R_3) = R_3.$$

Under the action of  $\iota$ , the  $(4, 4, 3; 3)$ -triangle groups will be sent to the triangle groups with braiding  $(2, 3, 4; 4)$

$$\begin{aligned}
\left\langle \iota(R_1), \iota(R_2)\iota(R_3) : \iota(R_2R_3) = \iota(R_3R_2), (\iota(R_1R_2))^{\frac{3}{2}} = (\iota(R_2R_1))^{\frac{3}{2}}, \right. \\
(\iota(R_1R_3))^2 = (\iota(R_3R_1))^2, \\
\left. (\iota(R_1R_2R_3R_2^{-1}))^2 = (\iota(R_2R_3R_2^{-1}R_1))^2 \right\rangle.
\end{aligned}$$

Recall the Mostow groups  $\Gamma(p, t)$  mentioned in [8, 10]. For Mostow groups, there exists a complex hyperbolic isometry  $J$  of order 3 so that  $R_{j+1} = JR_jJ^{-1}$  and  $R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$ . We could rewrite them as triangle groups with braiding  $(2, 3, 4; 4)$  as follows

$$\begin{aligned}
\langle R_1, R_2, J(R_1R_2)^{-1} : R_2J(R_1R_2)^{-1} = J(R_1R_2)^{-1}R_2, (R_1R_2)^{\frac{3}{2}} = (R_2R_1)^{\frac{3}{2}}, \\
(R_1J(R_1R_2)^{-1})^2 = (J(R_1R_2)^{-1}R_1)^2, \\
(R_1R_2J(R_1R_2)^{-1}R_2^{-1})^2 = (R_2J(R_1R_2)^{-1}R_2^{-1}R_1)^2 \rangle.
\end{aligned}$$

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